1203

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$c^{\flat}: V^{*}\otimes V \to V\otimes V^{*}$

$f\otimes v \longmapsto \sum (ev\otimes id\otimes id)(f\otimes c(v\otimes v_i)\otimes \alpha^{i})$

is invertible

(V, c)

Fomin-Kirillov alg. FK_m

$m = 3, 4, 5$ Hilbert series

$FK_n \subseteq \mathcal{B}(\mathbb{Q}^{n}, c^{\chi})$

affine racks $Aff(\mathbb{F}_q, T)$

$Aff(\mathbb{F}_4, \omega)$

$Aff(\mathbb{F}_5, 2)$

$Aff(\mathbb{F}_5, 3)$

$Aff(\mathbb{F}_7, 3)$ $Aff(\mathbb{F}_7, 5)$

9.1 Axioms and examples

$(\alpha_i)_{i\in I}$ Standard basis of $\mathbb{Z}^{I}$ $|I| < \infty$

Def. (Semi-Cartan graph) (Cartan scheme)

I non-empty set, $\mathcal{X}$ a non-empty set

$r: I\times \mathcal{X} \to \mathcal{X}$ $A: I\times I\times \mathcal{X} \to \mathbb{Z}$ maps

$r_i(X) = r(i, X)$ $a_{ij}^{X} = A(i, j, X)$

$A^{X} = (a_{ij}^{X})_{i,j\in I} \in \mathbb{Z}^{I\times I}$

$\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ is called a semi-Cartan graph if A^X is a Cartan matrix and $\qquad \mathcal{X} \to \mathcal{X}$, $X \mapsto r_i(X)$

(CG1) $\forall i \in I \quad r_i^2 = id_{\mathcal{X}}$

(CG2) $\forall i, j \quad X \in \mathcal{X} \qquad \underline{a_{ij}^X = a_{ij}^{r_i(X)}} \qquad \underline{+ s_i : \mathbb{Z}^I \to \mathbb{Z}^I}$

$A^X \quad A^{r_i(X)}$ have the same i-th row.

$|I|$: rank of $\mathcal{G}$ $\qquad X \to r_i(X) \; \vdots \; r_i \cdots r_k(X)$

elements of $\mathcal{X}$ $\qquad$ points of $\mathcal{G}$

elts of I $\qquad$ label of $\mathcal{G}$

$\underline{s_i^X \in \mathrm{Aut}(\mathbb{Z}^I)} \qquad s_i^X(\alpha_j) = \alpha_j - a_{ij}^X \alpha_i \qquad \forall j \in I$

$s_i^X(\alpha_i) = -\alpha_i$

r_i : permutation of $\mathcal{X}$

A : $(A^X)_{X \in \mathcal{X}}$

$(CG1) \Rightarrow \quad s_i^X = s_i^{r_i(X)} \qquad \alpha_j$

$I = \{1, 2\}$, $\mathcal{X} = \{X_1, X_2\}$, $\underline{r_1(X_1) = X_2,\ r_1(X_2) = X_1}$

$\underline{r_2 = id} \quad (r_1 = r_1)$

$A^{X_1} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}, \quad A^{X_2} = \begin{pmatrix} 2 & -1 \\ 4 & 2 \end{pmatrix} \qquad X_1 \underset{1,2}{\text{———}} X_2$

pf. (1) A^{X_1}, A^{X_2} Cartan matrix.

(2) $r_i^2 = id$

(3) $A^{X_1} \qquad A^{r_1(X_1)} = A^{X_2} \quad 1 \text{ 行}$

$\qquad\qquad A^{r_2(X_1)} = A^{X_1}$

$A^{X_2} \qquad A^{r_1(X_2)} = A^{X_1} \quad 1 \text{ 行}$

$\qquad\qquad A^{r_2(X_2)} = A^{X_2}$

Def. (exchange graph of $(\underline{\mathcal{X}}, r, I)$)

non-oriented

vertices $\longrightarrow$ elts of $\mathcal{X}$ $\qquad X \underset{i}{\text{———}} Y$

edges $\longrightarrow$ elts of I

X, Y are connected $\iff$ $X \neq Y$

$r_i(X) = Y$

$(r_i(Y) = X)$

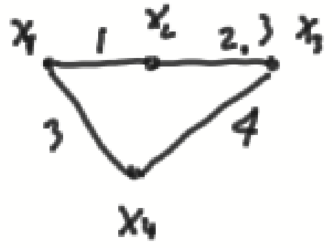
7. Example 2.1.4

$I = \{1, \dots, 5\}$ $\mathcal{X} = \{x_1, \dots, x_4\}$

$\underline{\sigma_1 = (12)}$, $\sigma_2 = (23)$ $\underline{\sigma_3 = (23)(14)}$, $\underline{\sigma_4 = (34)}$

$r_i(x_j) = x_{\sigma_i(j)}$ $1 \le i, j \le 4$ $r_5 = id$

(CG1) holds.



$r_1(\underline{x_1}) = x_{\sigma_1(1)} = x_2$

$\underline{r_2(x_2) = x_3}$

$r_3(x_2) = x_3$

$\underline{r_4(x_3) = x_4}$

$\underline{r_3(x_1) = x_4}$

SCG $\longrightarrow$ labeled exchange graph A^x

$\forall (\mathcal{X}, r, I)$ satisfying (CG1) $\Rightarrow \exists$ family of Cartan matrix A s.t. $\mathcal{G}(I, \mathcal{X}, r, A)$ is a SCG.

$\underline{A^x = A \quad \forall x \in \mathcal{X}}$.

<u>Def. (morphism of SCG)</u>

Def. (standard)

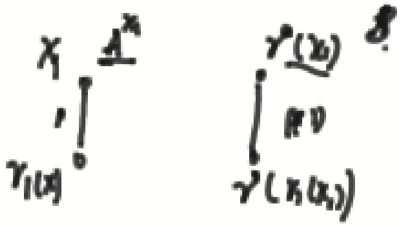
A SCG is standard if $A^x = A^y$ $\forall x, y \in \mathcal{X}$.

$\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$, $\mathcal{G}' = \mathcal{G}(J, \mathcal{Y}, t, B)$ are SCG.

$\underline{(\varphi, \gamma)}: \mathcal{G} \longrightarrow \mathcal{G}'$ is called a <u>morphism of SCG</u>

$\varphi: I \to J$, $\gamma: \mathcal{X} \to \mathcal{Y}$

$$\begin{tikzcd} I \times \mathcal{X} \arrow[r, "\varphi \times \gamma"] \arrow[d, "r"'] \arrow[dr, phantom, "\circlearrowleft"] & J \times \mathcal{Y} \arrow[d, "t"] \\ \mathcal{X} \arrow[r, "\gamma"'] & \mathcal{Y} \end{tikzcd} \qquad \begin{tikzcd} I \times I \times \mathcal{X} \arrow[r, "\varphi \times \varphi \times \gamma"] \arrow[d, "A"'] \arrow[dr, phantom, "\circlearrowleft"] & J \times J \times \mathcal{Y} \arrow[dl, "B"] \\ \mathbb{Z} & \end{tikzcd}$$



A

Def. 2.1.6 $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ is SCG

$\mathcal{Y} \subset \mathcal{X}$ non-empty $\underline{r_i(Y) \in \mathcal{Y}}$ $\forall i \in I,\ Y \in \mathcal{Y}$

$\mathcal{G}' = \mathcal{G}(I, \mathcal{Y}, r|_{(I \times \mathcal{Y})}, A|_{(I \times I \times \mathcal{Y})})$ is called a semi-Cartan subgraph of $\mathcal{G}$.

$(id, \underline{\mathcal{Y}}) : \mathcal{G}' \longrightarrow \mathcal{G}$ $\quad r$ = inclusion.

Def. (Connected) if $\exists$ no proper non-empty subset $\mathcal{Y} \subset \mathcal{X}$ s.t. $r_i(Y) \in \mathcal{Y}$ $\forall i \in I,\ Y \in \mathcal{Y}$.

r_i: $\underline{X \in \mathcal{X}}$

$\forall X \in \mathcal{X}$

$$\{ r_{i_1} \cdots r_{i_k}(X) \mid k \geq 0,\ i_1, \ldots, i_k \in I \}$$

is the only connected semi-Cartan subgraph containing X.

Connected component of $\mathcal{G}$ containing X.

Example 9.1.7. $\quad I = \{1, 2\}$ $\quad$ Then the $\underline{\text{connected component}}$ of $\mathcal{G}$ are